Critical aspects of equations when explored as a part-whole structure

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The aim of this paper is to present critical aspects that were identified when students explored equations as a part-whole structure with negative numbers included. Students in grades 3, 8 and 9 participated in a ”theoretical work”. Learning study was used as a research approach and learning activity theory constituted a guiding principle when designing research lessons. According to the analysis, five critical aspects were identified, regardless of grade. The critical aspects are: there is a relationship between all the numbers in an equation; two parts together equals a whole with the same *value*; what constitutes the parts and the whole, respectively; the same relationship can be formulated in four different ways; the whole can assume a lower value than the parts.

The aim of this paper is to present critical aspects that were identified when students explored relationships, as a part-whole structure, between numbers in equations (e.g. Schmittau, 2005). Accordingly, the critical aspects concern what students need to discern in order to learn how the numbers in an equation relate to each other (cf. Davydov, 2008). The equations consisted of additive structures (Vergnaud, 1982) and included negative numbers (integers). One challenge concerning teaching negative numbers may be that it is not straightforward to explore them empirically by quantities (Schubring, 2005). A reason for extending the numbers to negative numbers in this study was to challenge an assumption that subtraction tasks always lead to the difference consisting of a lower value than the minuend, and that addition tasks lead to the sum consisting of a higher value than the addends (Bishop et al., 2014). Another reason for extending the numbers was to challenge students not to just ”know” or ”see” the ”answers”. Challenging these assumptions was taken as a way to afford a strong focus on the relationship between the numbers in an equation rather than a focus on calculations.

One theme discussed in the literature is that when the focus is only on calculations based on rules and procedures, this may lead to students not being given an opportunity to reflect on mathematical structures beyond the rules

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and procedures at hand (Kilpatrick et al., 2001; Mason et al., 2012). Although students can solve routine tasks, based on rules and procedures on some occasions, challenges may occur when handling similar tasks in new situations when underlying structures are not discerned (Brown et al., 1988). Knowing about the inverses of addition and subtraction may be of importance, since subtraction is frequently declared in previous research as more difficult arithmetic than addition (e.g. Baroody, 1984; Brissiaud & Sander, 2010). Previous research has focused on addition and subtraction tasks, based on general structures as a part-whole structure (e.g. Carpenter et al., 1981). A part-whole structure can be depicted as the whole is built up by parts (Carpenter & Moser, 1982; Schmittau, 2005). Attributing a part-whole structure as a relationship between numbers by general symbols and not by specific values, means there is nothing to calculate, which may support students to focus on the general structures (Davydov, 2008). Focusing on general structures requires first and foremost to notice the structure and to analyse relationships between quantities and between numbers (Cai & Knuth, 2011; Kieran, 2018).

In this paper, we will answer the research question *What do students need to discern in order to master equations based on relationships between the numbers?* The findings discussed in the paper are in conjunction with two other articles dealing with contents closely related to each other (Andersson & Tuominen, in progress; Tuominen et al., 2018).

Methodology

The study was conducted with learning study as research approach, since it offers a basis for interventions building on systematical and iterative processes (Marton, 2015). This was of importance in order to identify what students need to discern in order to master equations based on relationships between the numbers. In order to address the research question, the notion of critical aspects was adopted (cf. Marton, 2015; Marton & Booth, 1997). *Critical aspects* is a core concept deriving from variation theory, a theory of learning. In order to learn the intended, in this case, relationships between numbers in equations (the object of learning), there are necessary aspects – critical aspects – to be discerned. Critical aspects are relational, which means there is an interconnection between the students, the object of learning, and in which ways the students experience the object of learning (Pang & Ki, 2016). In this study, we coordinated variation theory (mainly critical aspects) with learning activity theory (Davydov, 2008). Eriksson (2017) claims that the two theories are possible to combine in relation to their focus on what students need to learn and how they manage the content. Learning activity theory was used as a theoretical guiding principle when designing the research lessons, and it also provided a lens with regards to the focus when identifying the critical aspects (see Analysis section).

Learning activity theory suited our research interest since it addresses the development of students' consciousness and thinking regarding the theoretical knowledge accomplished through theoretical work. This kind of work is characterized by what Davydov (2008, p. 115) defines as "[...] contentful abstraction and generalization and theoretical concepts, taken as a unity [...]". Based on learning activity theory, a starting point is to introduce mathematical content based on general structures and subsequently to exemplify the content with specific numbers though still based on general structures (Davydov, 2008). In the case of our study, the theoretical work concerned relationships between numbers in equations, and, in order to capture and visualize the abstract properties, a learning model (see figure 1) was used (Davydov, 2008; Gorbov & Chudinova, 2000).

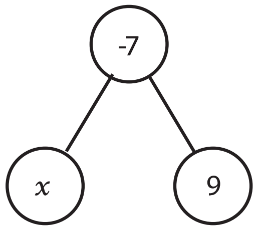

Figure 1. *The learning model used in our study, inspired by Davydov (2008)*

The intention with using the model was to enable students to identify the relationship between a whole and two parts, and how the same relationship can be formulated in four ways (cf. Davydov, 2008). This, in turn, may enable students to discern addition and subtraction as inverses (Greer, 2012).

Students, 149 in total, from grades 3, 8 and 9, attending compulsory school, participated in the study. The different grades were chosen based on the researchers' experiences as teachers. Initially there were two different projects, but as the researchers collaborated they noticed that the same tentative critical aspects were identified, regardless of grade. Due to this similarity, the researchers decided to collaborate in one research project. In total, nine video-recorded research lessons were conducted, while each student participated in one research lesson. According to the students' teachers, as well as findings from interviews with the students (see Tuominen et al., 2018), the students had no experiences of teaching based on relationships between numbers in equations, regardless of the students' different ages. None of the students in grade 3 had previous experiences of negative numbers as operands.

The data material consists of transcribed video recorded research lessons and audio recorded interviews, as well as the students' written expressions from the pre- and post-tests, and lessons. The pre- and post-tests were identical and consisted, largely of the same content, regardless of grade. Some of the students were interviewed after the pre- and post-tests. The intention was to explore how students grasped and managed the different tasks.

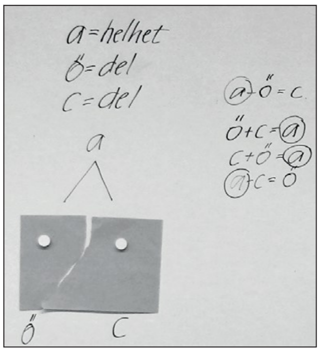

Figure 2. *A relationship between quantities, formulated by general symbols*

Initially, during the research lessons, a relationship between quantities was explored as a theoretical work. As depicted in figure 2, the whole *a* was divided into two parts by empirical material.

The relationship between the quantities was formulated with general symbols, in this case, with *a, c,* and *ö*. The intention with using general symbols initially was to enable students to focus on *the relationship*, not on something to calculate. The model in figure 2, constructed by a participating teacher and some of the participating students, was used as a learning model. Additionally, during each research lesson, the model in figure 1, constructed by Tuominen and Andersson, was used when the students explored relationships between *numbers* in equations, since it is not straightforward to explore negative numbers using *quantities*. This means that the teaching went from the general to the specific, though still as a theoretical work. For example, the students in grades 8 and 9 explored equations such as $-7 - x = 9$ and subsequently formulated the same relationship in further three ways (see figure 3). Based on a part-whole structure, ”*x*” and ”9” constitute the parts and ”-7” constitutes the whole; that is, ”-7” is built up by ”*x*” and ”9” (see the second and the third formulations in figure 3). This relationship applies regardless of the four formulations. In this relationship, the whole assumes a lower value than one of the parts, which is valid when negative numbers are included.

$$-7 - x = 9$$
$$x + 9 = -7$$
$$9 + x = -7$$
$$-7 - 9 = x$$

Figure 3. *A seemingly difficult equation, reformulated in three ways*

Discerning that one relationship can be formulated in several ways is about the need to discover that ”[...] any mathematical operation has an unambiguous structure [...]” (Davydov, 2008, p. 148). When students discern that the same relationship can be formulated in four ways, it may enable them to choose a more convenient equation to solve. This can be particularly advantageous when negative numbers are included.

Analysis

In order to identify critical aspects, an initial analytical question was formulated: *What are indications of critical aspects when exploring equations as a part-whole structure in theoretical work?* During the process of analysis, the focus was on what students expressed, orally and in writing. It showed that there was an interplay between the data and previous research concerning critical aspects (Marton, 2015; Pang & Ki, 2016), concerning relationships between numbers (e.g. Davydov, 2008; Schmittau, 2005), and concerning the theoretical work (Davydov, 2008). The analysis was conducted as follows. *First*, the video and audio recordings were listened to several times in order to distinguish expressions regarding relationships. *Second*, the recordings were transcribed verbatim. *Third*, the transcripts were read several times and students' expressions regarding equations and numbers, known or unknown, and mathematical symbols were highlighted. Also the students' written expressions from the pre- and post-tests, and lessons were analysed according to the same criteria. *Fourth*, whether, and in what way, the students expressed a relationship between the numbers in equations (cf. Davydov, 2008; Schmittau, 2005) was analysed, interpreted, and categorized. The guiding principle for this was how, in excerpts and elsewhere, there were qualitatively different ways (see Marton, 2015) of expressing a relationship. This process provided a basis for us to identify critical aspects regarding relationships. *Finally*, the analytical question *What do students need to discern in order to manage numbers in equations as relational?* was adopted. This was to support the process of an additional analysis, where excerpts from students' qualitative different expressions were compiled into five categories where each, ultimately, represented a critical aspect.

An example of how we interpreted different expressions is how excerpt 1 below was read as that Elli did not perceive that a relationship can be formulated in different ways, but rather that the two formulations were two separate equations with no connection between them. The excerpt was regarded as critical aspect number 4. Elli's expression in excerpt 2 below was placed in the critical aspect number 5, since it was interpreted that she did not discern the relationship between the numbers. Further, it was interpreted as if she supposed that equations with addition always lead to a sum with higher value than the addends. The expression in excerpt 2 differs qualitatively from the expression in excerpt 1. The five critical aspects are described in the Findings below.

Findings

The five critical aspects will be presented and exemplified below, through descriptions, excerpts, or figures. The critical aspects concern Davydov's (2008) discussion of students' consciousness and thinking regarding theoretical knowledge and work; in the case of this study, exploring equations as theoretical work. Based on the fact that critical aspects do not concern what students struggle with, but with what enables them to discern necessary aspects (Marton, 2015), there are examples in this section of when students discerned and when students did not discern necessary aspects. The order in which the critical aspects are presented does not imply the need for them to be discerned in that particular order.

1. There is a relationship between *all* the numbers in an equation

The critical aspect *there is a relationship between* all *the numbers in an equation* was manifested in different ways. An instance of when this aspect was possible to identify in the data was that several of the students, regardless of grade, did not express anything regarding the relationship between *all* the numbers. Rather, the students focused on the numbers in relation to the mathematical signs or the position of the numbers in an equation, without consideration of the other numbers included. One example of that is when students in the pre- and post-tests were supposed to formulate the equation $x-5=3$ in several ways. Sometimes, students placed the numbers as, for example, $5-3=x$, which resulted in a different relationship. In these two equations, ”x” consists of different values. In the analysis, it was interpreted that the students in the theoretical work were not conscious of, and did not experience, the importance of focusing on *all* the numbers, simultaneously.

2. Two parts together equals a whole with the same value

This critical aspect is based on a critical aspect identified in an analysis by Tuominen et al. (2018); *two quantities together (two parts) build up a third quantity (the whole) with the same ”value” as the two parts together.* The critical aspect identified in Tuominen et al. concerns quantities (in the form of volume). An equivalent critical aspect was also identified in this study. Because it concerns numbers in equations, instead of quantities, it is consequently formulated as *two parts together equals a whole with the same value.* In this analysis, we also formulated the same critical aspect, as *if one of the parts is taken away from the whole the other part is what remains.* Formulating the same critical aspect in different ways means that various perspectives are adopted, from parts or from a whole. An instance of when this critical aspect was possible to identify in the video data was when a teacher and students explored the relationships between quantities by using pieces of paper (figure 2) (cf. Davydov, 2008). During a research lesson in grade 3, students denominated the whole by *a* and the two parts by *c* and *ö* (the letters were chosen by the students) and one student suggested expressing the relationship as $c+c=a$. In the analysis, this was interpreted as the student not experiencing how the whole and the parts, the quantities *a*, *c* and *ö*, were related to each other. Another student in grade 3 expressed: ”if you take *ö* plus *c* it is equal to *a*” and later the same student expressed: ”*a* minus *ö* is equal *c*”. In the analysis, this was interpreted as the student discerning a relationship between the three numbers and how the same relationship can be formulated in two different ways.

3. What constitutes a whole and parts, respectively

Another critical aspect identified in the analysis, is *what constitutes a whole and parts, respectively*. An instance of when the critical aspect was found in the data was when students in grade 8, were encouraged in the post-test to mark "the whole" in four different equations. Below, there is a solution from one student, who marked the "answer" as the whole. The student's markings are depicted in bold and underlined (figure 4). Although the answer is the whole when it comes to addition, that is not the case when it comes to subtraction.

2 + 7 = **<u>9</u>**
(-7) + 2 = **<u>(-5)</u>**
2 – 7 = **<u>(-5)</u>**
(-7) – 2 = **<u>(-9)</u>**

Figure 4. *The whole experienced as the answer*
Note. The equations are reconstructed due to the poor quality of the original.

In the analysis, the markings in figure 4 indicate that the student did not experience the whole based on a part-whole structure. The student rather experienced the whole as the *answer* in the four different equations.

4. The same relationship can be formulated in four different ways

A further critical aspect is *the same relationship can be formulated in four different ways* (cf. Davydov, 2008; Schmittau, 2005). An instance of when this critical aspect was present in the data was when students in grade 3 were exploring a relationship between the numbers "x", "2" and "3". On the whiteboard, the teacher wrote two equations showing the same relationship as $x+3=2$ and $2-x=3$. The students were encouraged to identify the whole and the parts, supported by the model (figure 1) and to formulate the relationship in four ways.

Excerpt 1, grade 3

Teacher: What is the whole? [The teacher points to the whiteboard]
Elli: Ah, wait ... Are we talking about the first [equation] ?

The communication in excerpt 1 is an instance of this critical aspect. The example demonstrates indications of a student not expressing that the two equations represent the same relationship and thus, that the whole and the parts are the same regardless of the two shown equations. What the student expressed was interpreted in the analysis as the student rather experiencing the two separate equations as two different relationships.

5. The whole can assume a lower value than the parts

Finally, the critical aspect *the whole can assume a lower value than the parts* will be presented. An example of when the whole assumes a lower value than one of its parts is from the example above when students in grade 3 explored the relationship between the numbers in $x + 3 = 2$ and $2 - x = 3$. The students were encouraged to identify the relationship supported by the learning model (figure 1) and initially to identify the whole. What encouraged the students to formulate the relationship in four different ways was that the students were asked to find an appropriate operation in order to find the value of the unknown number ”*x*”. This is reproduced in excerpt 2 below.

Excerpt 2, grade 3

Teacher:	What is the whole? [The teacher points at the whiteboard] [...]
Elli:	We did so that we, the whole ... we thought the whole was five, since we turned it around so it instead became two plus three equals as something. We somehow turned direction [change the value of the whole and the parts]. We turned a little.

Excerpt 2 is an instance of the critical aspect. Based on Elli's expression, the students in the group seemed to be aware that they had altered the equation. In the analysis, this was interpreted as if the students supposed that equations with addition always lead to a sum with a higher value than the addends. Similar examples were identified with students in grade 9. Further, it was interpreted that students did not discern a relationship. In another example, Ali, a student in grade 3, expressed: ”And that [points at *x*], 'one-minus' plus three is equal to two”. In the analysis, this was interpreted as if the student discerned that a whole can assume a lower value than the parts, i.e. that the student discerned this critical aspect.

Summary of findings and concluding discussion

In the analysis, five critical aspects were identified when students participated in teaching inspired by learning activity theory (see Davydov, 2008). Previous research regarding relationships as a part-whole structure (e.g. Davydov, 2008; Schmittau, 2005), inspired us when exploring equations in order to identify critical aspects. So far, we have not found previous research regarding a part-whole structure where the whole assumes a lower value than the parts. For that reason, the critical aspect number 5, presented above, emerged in this empirical study. According to variation theory, an assumption is that in order to distinguish ”new aspects”, the teacher needs to take the differences between respective critical aspects into account, when designing the teaching (cf. Marton, 2015). Hence, five critical aspects could be perceived as too many. However, we argue there are justifiably five critical aspects concerning relationships and equations including negative numbers. One reason for including all five critical aspects may be that none of the students had previous experiences of being taught general mathematical structures. The assumption can be supported by Pang and Ki's (2016) emphasis that there is an interconnection between the students, the object of learning, and the ways in which students experience the object of learning.

Cai and Knuth (2011) claim there is a need for analysing relationships between quantities and between numbers and noticing structures. We argue it is not enough to *notice* structures. First, there is a need to experience that there *is* a relationship between the numbers in an equation. We also claim, that it is not enough to experience a relationship between *two* of the numbers, for example, between $x-5$ in the equation $x-5=3$. The numbers are not solitaires, which can be manipulated one by one when exploring equations as a relationship. Drawing on this, we state that students benefit from discerning the relationship between *all* the numbers in an equation.

The critical aspect *two parts together equals a whole with same value as the two parts together* is based on a part-whole structure, which in turn requires that students need to discern what constitute the parts and the whole, respectively, and further, simultaneously. When students are not already familiar with teaching based on general structures and the intention is to enable students to focus on a structure, there may be a need for initially exploring equations with quantities and using general symbols (cf. Davydov, 2008). Without anything to calculate, it may enable students to identify, for example, a part-whole structure (cf. Schmittau, 2005).

When the critical aspect *the same relationship can be formulated in four different ways* was explored, the learning model (figure 1) came to play an important role for some of the students. The learning model functioned as a mediating tool and enabled students to identify and formulate all four equations, reflecting the same relationship (cf. Davydov, 2008; Gorbov & Chudinova, 2000). Nevertheless, some of the students did not experience that $x+3=2$ and $2-x=3$ concern the same relationship. Although the critical aspects are many, they are intertwined. Maybe the critical aspect *the whole can assume a lower value than the parts*, stands out from the others.

When analysing research lessons, it became clear that the choice of values in equations was important. When the numbers in the equations were too simple, the students did not focus on general structures as relationships between numbers, and thus the part-whole structure. Students rather tried to solve the equations as they were used to doing – by rules and procedures – which was not always advantageous, since some students did not remember the rules and procedures (cf. Brown et al., 1988). Further, when the values of the numbers were too simple, there was no need for a learning model, nor the four formulations. When not challenging the students by using negative numbers, the whole always assumes a higher value than one or all of the parts. This can lead to an undesirable experience (Bishop et al., 2014). For that reason, negative numbers were of importance in this study. Visualizing a part-whole structure and that one relationship can be formulated in four different ways, has been shown to be powerful in this paper, not the least when negative numbers are included in equations (figure 3). Further, when students are proficient in addition and subtraction as inverses and in additive structures, it may enable them to choose an appropriate (for them) and convenient operation when solving equations (cf. Greer, 2012; Vergnaud, 1982).

There are limitations in the study. For example, the students participated in a teaching context concerning a content (general mathematical structures), which were unfamiliar. Despite that, we argue, there are implications for teaching. It is worth changing teaching from merely focusing on calculations based on rules and procedures, into teaching based on general structures. One reason is that the older students mostly focused on calculations based on rules and procedures, even despite (see Uziel & Amit, 2019) having attended school for many years. Another reason is that several of the younger students solved equations such as $x+3=2$, supported by the learning model even though they had no experience of exploring equations as a part-whole structure or experience of negative numbers as operands. Although critical aspects are relational, teachers can use the critical aspects identified in our study as a starting point when planning lessons when teaching concerns relationships between numbers in equations. In order to determine whether the critical aspects need to be discerned in a specific order, more research is required. This can be seen as another limitation in this study.

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